

Completeness of eigenfunctions of Sturm-Liouville problems with discontinuities at three points

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ABSTRACT.

In this work, we study discontinuous Sturm-Liouville type problems with eigenparameter dependent boundary condition and transmission conditions at three interior points. A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of such a problem coincide with those of A . We show that the eigenvalues of the problem are analytically simple, and the eigenfunctions of A are complete in H .

1 Introduction

It is well-known that many topics in mathematical physics require the investigation of eigenvalues and eigenfunctions of Sturm-Liouville type boundary value problems. In recent years, more and more researches are interested in the discontinuous Sturm-Liouville problems with eigenparameter-dependent boundary conditions (see [1 – 4]). The literature on this subject is voluminous and we refer to [5 – 10]. Various physics applications of this kind problem are found in many literatures, including some boundary value problem with transmission conditions that arise in the theory of heat and mass transfer (see [2, 10, 11]). The study of the structure of the solution in the matching region of the layer with the basis solution in the plate leads to consideration of an eigenvalue problem for a second order differential operator with piecewise continuous coefficients and transmission conditions [12].

Sturm-Liouville problems with transmission conditions have been studied by many authors (see [3, 4, 8, 13]). Adjoint and self-adjoint boundary value problems with interface conditions have been studied in [14, 15]. Such problems with point interactions are also studied in [16].

In this study, we also deal with the class of problems (1) – (9), by means of

a combination of the methods [5],[13],[17] and[18]. In Section 1, a self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of the problem (1) – (9) coincide with those of A . In Section 2, we prove that the eigenvalues of the problem (1) – (9) are analytically simple. In Section 3, we prove that the eigenfunctions of A are complete in H . Note that each eigenfunction of the original problem and a real number.

In this study, we consider a discontinuous eigenvalue problem which consists of Sturm-Liouville equation

$$\tau u := \left(-p(x) u'(x) \right)' + q(x)u(x) = \lambda u(x) \quad (1)$$

on $I = [-1, h_1) \cup (h_1, h_2) \cup (h_2, h_3) \cup (h_3, 1]$, where $p(x) = p_1^2$ for $x \in [-1, h_1)$; $p(x) = p_2^2$, for $x \in (h_1, h_2)$, $p(x) = p_3^2$, for $x \in (h_2, h_3)$ and $p(x) = p_4^2$, for $x \in (h_3, 1]$; p_1, p_2, p_3, p_4 are nonzero real constants, $q(x) \in L^1(I, \mathbb{R})$ and $\lambda \in \mathbb{C}$ is the eigenparameter; with the boundary condition

$$\tau_1 u := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \quad (2)$$

the eigenparameter-dependent boundary condition

$$\tau_2 u := \lambda \left[\beta_1' u(1) - \beta_2' u'(1) \right] + \left[\beta_1 u(1) - \beta_2 u'(1) \right] = 0, \quad (3)$$

and the transmission conditions

$$\tau_3 u := u(h_1 + 0) - \alpha_3 u(h_1 - 0) - \beta_3 u'(h_1 - 0) = 0, \quad (4)$$

$$\tau_4 u := u'(h_1 + 0) - \alpha_4 u(h_1 - 0) - \beta_4 u'(h_1 - 0) = 0, \quad (5)$$

$$\tau_5 u := u(h_2 + 0) - \alpha_5 u(h_2 - 0) - \beta_5 u'(h_2 - 0) = 0, \quad (6)$$

$$\tau_6 u := u'(h_2 + 0) - \alpha_6 u(h_2 - 0) - \beta_6 u'(h_2 - 0) = 0, \quad (7)$$

$$\tau_7 u := u(h_3 + 0) - \alpha_7 u(h_3 - 0) - \beta_7 u'(h_3 - 0) = 0, \quad (8)$$

$$\tau_8 u := u'(h_3 + 0) - \alpha_8 u(h_3 - 0) - \beta_8 u'(h_3 - 0) = 0, \quad (9)$$

where the coefficients α_i, β_i and β_j' ($i = \overline{1, 8}, j = 1, 2$) are real numbers. Throughout this paper, we assume that

$$\theta = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} > 0, \gamma = \begin{vmatrix} \alpha_5 & \beta_5 \\ \alpha_6 & \beta_6 \end{vmatrix} > 0, \xi = \begin{vmatrix} \alpha_7 & \beta_7 \\ \alpha_8 & \beta_8 \end{vmatrix} > 0,$$

$$\rho = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0,$$

and $|\alpha_1| + |\alpha_2| \neq 0$.

2 Operator formulation

The relation between a symmetric linear operator A defined in a suitable Hilbert space H and the problem (1) – (9) has been introduced in [13]. Here, we repeat the definition and prove that the operator A is self-adjoint, not only symmetric.

We define the inner product in $L^2(I)$ as

$$\langle f, g \rangle_1 = \frac{1}{p_1^2} \int_{-1}^{h_1} f_1 \overline{g_1} + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f_2 \overline{g_2} + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^1 f_3 \overline{g_3} + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_2}^1 f_4 \overline{g_4}, \quad \forall f, g \in L^2(I),$$

where

$$f_1(x) := \begin{cases} f(x), & x \in [-1, h_1), \\ \lim_{x \rightarrow h_1-0} f(x), & x = h_1, \end{cases} \quad f_2(x) := \begin{cases} \lim_{x \rightarrow h_1+0} f(x), & x = h_1, \\ f(x), & x \in (h_1, h_2), \\ \lim_{x \rightarrow h_2-0} f(x), & x = h_2, \end{cases}$$

$$f_3(x) := \begin{cases} \lim_{x \rightarrow h_2+0} f(x), & x = h_2, \\ f(x), & x \in (h_2, h_3), \\ \lim_{x \rightarrow h_3-0} f(x), & x = h_3, \end{cases} \quad f_4(x) := \begin{cases} \lim_{x \rightarrow h_3+0} f(x), & x = h_3, \\ f(x), & x \in (h_3, 1] \end{cases}$$

It is easy to verify that $(L^2(I), \langle \cdot, \cdot \rangle_1)$ is a Hilbert space. For simplicity, it is denoted by H_1 .

The inner product in $H := H_1 \oplus \mathbb{C}$ is defined by

$$\langle F, G \rangle = \langle f, g \rangle_1 + \frac{1}{\rho \theta \gamma \xi} h \overline{k}$$

for $F = (f(x), h), G = (g(x), k) \in H$, where $f, g \in H_1, h, k \in \mathbb{C}$.

We define the operator A in H as follows:

$$D(A) := \left\{ (f(x), h) \in H \mid f_1, f'_1 \in AC_{loc}((-1, h_1)), f_2, f'_2 \in AC_{loc}((h_1, h_2)), \right. \\ \left. f_3, f'_3 \in AC_{loc}((h_2, h_3)), f_4, f'_4 \in AC_{loc}((h_3, 1)), \tau f \in H_1, \tau_1 f = \tau_3 f = \tau_4 f = \tau_5 f = \tau_6 f, \right. \\ \left. h = \beta_1 f(1) - \beta_2 f'(1), \right\}$$

$$AF = \left(\tau f, - \left(\beta_1 f(1) - \beta_2 f'(1) \right) \right) \text{ for } F = \left(f, \beta_1 f(1) - \beta_2 f'(1) \right) \in D(A).$$

Note that by our assumption on $q(x)$ and Theorem 3.2 in [19], for each $(f, h) \in D(A)$, f_1, f'_1 are continuous on $[-1, h_1]$, f_2, f'_2 are continuous on $[h_1, h_2]$, f_3, f'_3 are continuous on $[h_2, h_3]$ and f_4, f'_4 are continuous on $[h_3, 1]$. For simplicity, for $(f, h) \in D(A)$, set

$$N(f) = \beta_1 f(1) - \beta_2 f'(1), \quad N'(f) = \beta'_1 f(1) - \beta'_2 f'(1).$$

So, we can study the problem (1) – (9) in H by considering the operator equation $AF = \lambda F$. Obviously, we have

Lemma 1.1 *The eigenvalues of the boundary value problem (1) – (9) coincide with those of A , and its eigenfunctions are the first components of the corresponding eigenfunctions of A .*

Lemma 1.2 *The domain $D(A)$ is dense in H .*

Proof. Suppose that $F \in H$ is orthogonal to all $G \in D(A)$ with respect to the inner product $\langle \cdot, \cdot \rangle$, where $F = (f(x), h)$, $G = (g(x), k)$. Let \tilde{C}_0^∞ denote the set of functions

$$\phi(x) = \begin{cases} \varphi_1(x), & x \in [-1, h_1), \\ \varphi_2(x), & x \in (h_1, h_2), \\ \varphi_3(x), & x \in (h_2, h_3), \\ \varphi_4(x), & x \in (h_3, 1], \end{cases}$$

where $\varphi_1(x) \in C_0^\infty[-1, h_1)$, $\varphi_2(x) \in C_0^\infty(h_1, h_2)$, $\varphi_3(x) \in C_0^\infty(h_2, h_3)$ and $\varphi_4(x) \in C_0^\infty(h_3, 1]$. Since $\tilde{C}_0^\infty \oplus 0 \subset D(A)$ ($0 \in \mathbb{C}$), any $U = (u(x), 0) \in \tilde{C}_0^\infty \oplus 0$ is orthogonal to F , namely,

$$\begin{aligned} \langle F, U \rangle &= \frac{1}{p_1^2} \int_{-1}^{h_1} f(x) \overline{u(x)} dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f(x) \overline{u(x)} dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} f(x) \overline{u(x)} dx \\ &\quad + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 f(x) \overline{u(x)} dx = \langle f, u \rangle_1. \end{aligned}$$

This implies that $f(x)$ is orthogonal to \tilde{C}_0^∞ in H_1 and hence vanishes. So, $\langle F, G \rangle = \frac{1}{\rho \theta \gamma \xi} h \bar{k} = 0$. Thus, $h = 0$ since $k = N'(g)$ can be chosen arbitrarily. So, $F = (0, 0)$. Therefore, $D(A)$ is dense in H . ■

Theorem 1.1. *The linear operator A is self-adjoint in H .*

Proof. For all $F, G \in D(A)$, (2) implies that $f(-1) \bar{g}'(-1) - f'(-1) \bar{g}(-1) = 0$, and direct calculations using (4) and (5) then yield that

$$\begin{aligned} \langle AF, G \rangle &= \langle F, AG \rangle + W(f, \bar{g}; h_1 - 0) - W(f, \bar{g}; -1) + \frac{1}{\theta} W(f, \bar{g}; h_2 - 0) - \\ &\quad \frac{1}{\theta} W(f, \bar{g}; h_1 + 0) + \frac{1}{\theta \gamma} W(f, \bar{g}; h_3 - 0) - \frac{1}{\theta \gamma} W(f, \bar{g}; h_2 + 0) + \\ &\quad \frac{1}{\theta \gamma \xi} W(f, \bar{g}; 1) - \frac{1}{\theta \gamma \xi} W(f, \bar{g}; h_3 + 0) - \frac{1}{\rho \theta \gamma \xi} \left(N(f) \overline{N'(g)} - N'(f) \overline{N(g)} \right) \\ &= \langle F, AG \rangle, \end{aligned}$$

where $(f, g; x)$ denotes the Wronskians $f(x) g'(x) - f'(x) g(x)$. So, A is symmetric.

It remains to show that if $\langle AF, W \rangle = \langle F, U \rangle$ for all $F = (f, N'(f)) \in D(A)$, then $W \in D(A)$ and $AW = U$, where $W = (w(x), h)$ and $U = (u(x), k)$, i.e., (i) $w_1, w'_1 \in AC_{loc}((-1, h_1))$, $w_2, w'_2 \in AC_{loc}((h_1, h_2))$, $w_3, w'_3 \in AC_{loc}((h_2, h_3))$, $w_4, w'_4 \in AC_{loc}((h_3, 1))$ and $\tau w \in H_1$; (ii) $h = N'(w) = \beta'_1 w(1) - \beta'_2 w(1)$; (iii) $\tau_1 w = \tau_3 w = \tau_4 w = \tau_5 w = \tau_6 w = \tau_7 w = \tau_8 w = 0$; (iv) $u(x) = \tau w$; (v) $k = -N(w) = -\beta_1 w(1) + \beta_2 w(1)$.

For all $F \in \tilde{C}_0^\infty \oplus 0 \subset D(A)$, we obtain

$$\begin{aligned} & \frac{1}{p_1^2} \int_{-1}^{h_1} (\tau f) \bar{w} dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} (\tau f) \bar{w} dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} (\tau f) \bar{w} dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 (\tau f) \bar{w} dx \\ &= \frac{1}{p_1^2} \int_{-1}^{h_1} f \bar{u} dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f \bar{u} dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} f \bar{u} dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 f \bar{u} dx, \end{aligned}$$

namely, $\langle \tau f, w \rangle_1 = \langle f, u \rangle_1$. Hence, by standart Sturm-Liouville theory, (i) and (iv) hold. By (iv), the equation $\langle AF, W \rangle = \langle F, U \rangle$, $\forall F \in D(A)$, becomes

$$\begin{aligned} & \frac{1}{p_1^2} \int_{-1}^{h_1} (\tau f) \bar{w} dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} (\tau f) \bar{w} dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} (\tau f) \bar{w} dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 (\tau f) \bar{w} dx - \frac{N(f) \bar{h}}{\rho \theta \gamma} \\ &= \frac{1}{p_1^2} \int_{-1}^{h_1} f (\tau \bar{w}) dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f (\tau \bar{w}) dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} f (\tau \bar{w}) dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 f (\tau \bar{w}) dx + \frac{N'(f) \bar{k}}{\rho \theta \gamma \xi}. \end{aligned}$$

So,

$$\langle \tau f, w \rangle_1 = \langle f, \tau w \rangle_1 + \frac{N'(f) \bar{k} + N(f) \bar{h}}{\rho \theta \gamma \xi}.$$

However,

$$\begin{aligned} \langle \tau f, w \rangle_1 &= \frac{1}{p_1^2} \int_{-1}^{h_1} (-p_1^2 f'' + q(x) f) \bar{w} dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} (-p_2^2 f'' + q(x) f) \bar{w} dx + \\ & \quad \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} (-p_3^2 f'' + q(x) f) \bar{w} dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 (-p_4^2 f'' + q(x) f) \bar{w} dx \\ &= \frac{1}{p_1^2} \int_{-1}^{h_1} f (\tau \bar{w}) dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f (\tau \bar{w}) dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} f (\tau \bar{w}) dx + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 f (\tau \bar{w}) dx \\ &+ W(f, \bar{w}; h_1 - 0) - W(f, \bar{w}; -1) + \frac{1}{\theta} W(f, \bar{w}; h_2 - 0) - \frac{1}{\theta} W(f, \bar{w}; h_1 + 0) + \\ & \quad \frac{1}{\theta \gamma} W(f, \bar{w}; h_3 - 0) - \frac{1}{\theta \gamma} W(f, \bar{w}; h_2 + 0) + \frac{1}{\theta \gamma \xi} W(f, \bar{w}; 1) - \frac{1}{\theta \gamma \xi} W(f, \bar{w}; h_3 + 0) \\ &= \langle f, \tau w \rangle_1 + W(f, \bar{w}; h_1 - 0) - W(f, \bar{w}; -1) + \frac{1}{\theta} W(f, \bar{w}; h_2 - 0) - \frac{1}{\theta} W(f, \bar{w}; h_1 + 0) \\ &+ \frac{1}{\theta \gamma} W(f, \bar{w}; h_3 - 0) - \frac{1}{\theta \gamma} W(f, \bar{w}; h_2 + 0) + \frac{1}{\theta \gamma \xi} W(f, \bar{w}; 1) - \frac{1}{\theta \gamma \xi} W(f, \bar{w}; h_3 + 0). \end{aligned}$$

Hence,

$$\begin{aligned}
\frac{N'(f)\bar{k} + N(f)\bar{h}}{\rho\theta\gamma\xi} &= W(f, \bar{w}; h_1 - 0) - W(f, \bar{w}; -1) + \frac{1}{\theta}W(f, \bar{w}; h_2 - 0) \\
&\quad - \frac{1}{\theta}W(f, \bar{w}; h_1 + 0) + \frac{1}{\theta\gamma}W(f, \bar{w}; h_3 - 0) - \frac{1}{\theta\gamma}W(f, \bar{w}; h_2 + 0) \\
&\quad + \frac{1}{\theta\gamma\xi}W(f, \bar{w}; 1) - \frac{1}{\theta\gamma\xi}W(f, \bar{w}; h_2 + 0) \\
&= \left(f(h_1 - 0)\bar{w}'(h_1 - 0) - f'(h_1 - 0)\bar{w}(h_1 - 0) \right) - \left(f(-1)\bar{w}'(-1) \right. \\
&\quad \left. - f'(-1)\bar{w}(-1) \right) + \frac{1}{\theta} \left(f(h_2 - 0)\bar{w}'(h_2 - 0) - f'(h_2 - 0)\bar{w}(h_2 - 0) \right) \\
&\quad - \frac{1}{\theta} \left(f(h_1 + 0)\bar{w}'(h_1 + 0) - f'(h_1 + 0)\bar{w}(h_1 + 0) \right) \\
&\quad + \frac{1}{\theta\gamma} \left(f(h_3 - 0)\bar{w}'(h_3 - 0) - f'(h_3 - 0)\bar{w}(h_3 - 0) \right) \\
&\quad - \frac{1}{\theta\gamma} \left(f(h_2 + 0)\bar{w}'(h_2 + 0) - f'(h_2 + 0)\bar{w}(h_2 + 0) \right) + \frac{1}{\theta\gamma\xi} \left(f(1)\bar{w}'(1) \right. \\
&\quad \left. - f'(1)\bar{w}(1) \right) - \frac{1}{\theta\gamma\xi} \left(f(h_3 + 0)\bar{w}'(h_3 + 0) - f'(h_3 + 0)\bar{w}(h_3 + 0) \right). \tag{8}
\end{aligned}$$

By Naimark's Patching Lemma [20], there is an $F \in D(A)$ such that $f(-1) = f'(-1) = f(h_1 - 0) = f'(h_1 - 0) = f(h_1 + 0) = f'(h_1 + 0) = f(h_2 - 0) = f'(h_2 - 0) = f(h_2 + 0) = f'(h_2 + 0) = f(h_3 - 0) = f'(h_3 - 0) = f(h_3 + 0) = f'(h_3 + 0) = 0$, $f(1) = \beta_2$ and $f'(1) = \beta_1$. For such an F , $N'(f) = 0$. Thus, from (8) we obtain $h = \beta_1 w(1) - \beta_2 w'(1)$. Namely, (ii) holds. Similarly, one proves (v).

It remains to show that (iii) holds. Choose $F \in D(A)$ so that $f(1) = f'(1) = f(h_1 - 0) = f'(h_1 - 0) = f(h_2 - 0) = f'(h_2 - 0) = f(h_3 - 0) = f'(h_3 - 0) = 0$, $f(-1) = \alpha_2$ and $f'(-1) = -\alpha_1$. $N'(f) = N(f) = 0$. From (8), we get $\alpha_1 w(-1) + \alpha_2 w'(-1) = 0$. Let $F \in D(A)$ satisfies $f(1) = f'(1) = f(-1) = f'(-1) = f(h_1 + 0) = f(h_2 + 0) = f(h_3 + 0) = 0$, $f(h_1 - 0) = -\beta_3$, $f(h_2 - 0) = -\beta_5$, $f(h_3 - 0) = -\beta_7$, $f'(h_1 - 0) = \alpha_3$, $f'(h_2 - 0) = \alpha_5$, $f'(h_3 - 0) = \alpha_7$, $f'(h_1 + 0) = \theta$, $f'(h_2 + 0) = \gamma$ and $f'(h_3 + 0) = \xi$. Then $N(f) = N'(f) = 0$. By (8), we have $w(h_1 + 0) = \alpha_3 w(h_1 - 0) + \beta_3 w'(h_1 - 0)$, $w(h_2 + 0) = \alpha_5 w(h_2 - 0) + \beta_5 w'(h_2 - 0)$ and $w(h_3 + 0) = \alpha_7 w(h_3 - 0) + \beta_7 w'(h_3 - 0)$. Finally, choose $F \in D(A)$ so that $f(1) = f'(1) = f(-1) = f'(-1) = f'(h_1 + 0) = f'(h_2 + 0) = f'(h_3 + 0) = 0$, $f(h_1 - 0) = \beta_4$, $f'(h_1 - 0) = -\alpha_4$, $f(h_2 - 0) = \beta_6$, $f'(h_2 - 0) = -\alpha_6$, $f(h_3 - 0) = \beta_8$, $f'(h_3 - 0) = -\alpha_8$, $f(h_1 + 0) = \theta$, $f(h_2 + 0) = \gamma$ and $f(h_3 + 0) = \xi$. Then $N(f) = N'(f) = 0$. From (8), we obtain $w'(h_1 + 0) = \alpha_4 w(h_1 - 0) + \beta_4 w'(h_1 - 0)$, $w'(h_2 + 0) = \alpha_6 w(h_2 - 0) + \beta_6 w'(h_2 - 0)$ and $w'(h_3 + 0) = \alpha_8 w(h_3 - 0) + \beta_8 w'(h_3 - 0)$. ■

Corollary 1.1 *The eigenvalues of (1) – (9) are real, and if λ_1 and λ_2 are two different eigenvalues of (1) – (9), then the corresponding eigenfunctions*

$f(x)$ and $g(x)$ are orthogonal in the sense of

$$\begin{aligned} & \frac{1}{p_1^2} \int_{-1}^{h_1} f \bar{g} + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} f \bar{g} + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} f \bar{g} + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 f \bar{g} + \\ & \frac{1}{\rho \theta \gamma \xi} \left(\beta_1' f(1) - \beta_2' f'(1) \right) \left(\beta_1' \bar{g}(1) - \beta_2' \bar{g}'(1) \right) = 0. \end{aligned}$$

3 Simplicity of eigenvalues

We consider the initial-value problem

$$\begin{cases} -p_1^2 u''(x) + q(x)u(x) = \lambda u(x), & x \in [-1, h_1), \\ u(-1) = \alpha_2, \quad u'(-1) = -\alpha_1. \end{cases}$$

In terms of existence and uniqueness in ordinary differential equation theory, the initial-value problem has a unique solution $\varphi_1(x, \lambda)$ for every $\lambda \in \mathbb{C}$. Similarly, the initial-value problem

$$\begin{cases} -p_2^2 u''(x) + q(x)u(x) = \lambda u(x), & x \in (h_1, h_2), \\ u(h_1) = \alpha_3 \varphi_1(h_1, \lambda) + \beta_3 \varphi_1'(h_1, \lambda), \\ u'(h_1) = \alpha_4 \varphi_1(h_1, \lambda) + \beta_4 \varphi_1'(h_1, \lambda) \end{cases}$$

has a unique solution $\varphi_2(x, \lambda)$ for every $\lambda \in \mathbb{C}$. The initial-value problem

$$\begin{cases} -p_3^2 u''(x) + q(x)u(x) = \lambda u(x), & x \in (h_2, h_3), \\ u(h_2) = \alpha_5 \varphi_2(h_2, \lambda) + \beta_5 \varphi_2'(h_2, \lambda), \\ u'(h_2) = \alpha_6 \varphi_2(h_2, \lambda) + \beta_6 \varphi_2'(h_2, \lambda) \end{cases}$$

has a unique solution $\varphi_3(x, \lambda)$ for every $\lambda \in \mathbb{C}$. Similarly, the initial-value problem

$$\begin{cases} -p_4^2 u''(x) + q(x)u(x) = \lambda u(x), & x \in (h_3, 1], \\ u(h_3) = \alpha_7 \varphi_3(h_3, \lambda) + \beta_7 \varphi_3'(h_3, \lambda), \\ u'(h_3) = \alpha_8 \varphi_3(h_3, \lambda) + \beta_8 \varphi_3'(h_3, \lambda) \end{cases}$$

has a unique solution $\varphi_4(x, \lambda)$ for every $\lambda \in \mathbb{C}$. For each given $x \in [-1, h_1)$, $\varphi_1(x, \lambda)$ is an entire function of λ ; for every $x \in (h_1, h_2)$, $\varphi_2(x, \lambda)$ is an entire function of λ ; for every $x \in (h_2, h_3)$, $\varphi_3(x, \lambda)$ is an entire function of λ and for every $x \in (h_3, 1]$, $\varphi_4(x, \lambda)$ is an entire function of λ .

Now we define a function $\phi(x, \lambda)$ on $x \in [-1, h_1) \cup (h_1, h_2) \cup (h_2, h_3) \cup (h_3, 1]$ by

$$\phi(x, \lambda) = \begin{cases} \varphi_1(x, \lambda), & x \in [-1, h_1), \\ \varphi_2(x, \lambda), & x \in (h_1, h_2), \\ \varphi_3(x, \lambda), & x \in (h_2, h_3), \\ \varphi_4(x, \lambda), & x \in (h_3, 1]. \end{cases}$$

Obviously $\phi(x, \lambda)$ satisfies (1), (2) and (4)–(9). Similarly, we define the function

$$\chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), & x \in [-1, h_1], \\ \chi_2(x, \lambda), & x \in (h_1, h_2), \\ \chi_3(x, \lambda), & x \in (h_2, h_3), \\ \chi_4(x, \lambda), & x \in (h_3, 1], \end{cases}$$

which satisfies (1), (3) – (9).

The Wronskian $W(\varphi_i(x, \lambda), \chi_i(x, \lambda))$ ($i = 1, 2, 3, 4$) are independent of the variable x . Let $w_i(\lambda) = W(\varphi_i(x, \lambda), \chi_i(x, \lambda))$ and $w(\lambda) = w_1(\lambda)$, and then we obtain $w_2(\lambda) = \theta w(\lambda)$, $w_3(\lambda) = \theta_\gamma w(\lambda)$ and $w_4(\lambda) = \theta_\gamma \xi w(\lambda)$.

Lemma 2.1 [10] *The eigenvalues of the problem (1) – (9) coincide with the zeros of the entire function $w(\lambda)$.*

Definition 2.1 *The analytic multiplicity of an eigenvalue λ of (1) – (9) is its order as a root of the characteristic equation $w(\lambda) = 0$. The geometric multiplicity of an eigenvalue is the dimension of its eigenspace, i.e., the number of its linearly independent eigenfunctions.*

For convenience, set $\phi = \phi(x, \lambda)$, $\chi_{1\lambda} = \frac{\partial \chi_1}{\partial \lambda}$, $\chi'_{1\lambda} = \frac{\partial \chi'_1}{\partial \lambda}$, etc.

Theorem 2.1 *The eigenvalues of (1) – (9) are analytically simple.*

Proof. Let $\lambda = s + it$, where $s, t \in \mathbb{R}$ and $i^2 = -1$. We differentiate the equation $\tau\chi = \lambda\chi$ with respect to λ and have

$$\tau\chi_\lambda = \lambda\chi_\lambda + \chi.$$

By integration by parts, we get

$$\begin{aligned} \langle \tau\chi_\lambda, \phi \rangle_1 - \langle \chi_\lambda, \tau\phi \rangle_1 &= \left(\chi_{1\lambda} \overline{\varphi_1}' - \chi'_{1\lambda} \overline{\varphi_1} \right) \Big|_{-1}^{h_1} + \frac{1}{\theta} \left(\chi_{2\lambda} \overline{\varphi_2}' - \chi'_{2\lambda} \overline{\varphi_2} \right) \Big|_{h_1}^{h_2} + \\ &+ \frac{1}{\theta_\gamma} \left(\chi_{3\lambda} \overline{\varphi_3}' - \chi'_{3\lambda} \overline{\varphi_3} \right) \Big|_{h_2}^{h_3} + \frac{1}{\theta_\gamma \xi} \left(\chi_{4\lambda} \overline{\varphi_4}' - \chi'_{4\lambda} \overline{\varphi_4} \right) \Big|_{h_3}^1. \end{aligned} \quad (9)$$

Substituting $\tau\chi_\lambda = \lambda\chi_\lambda + \chi$ and $\tau\phi = \lambda\phi$ into the left side of (9), we have

$$\lambda \langle \chi_\lambda, \phi \rangle_1 + \langle \chi, \phi \rangle_1 - \langle \chi_\lambda, \lambda\phi \rangle_1 = \langle \chi, \phi \rangle_1 + 2it \langle \chi_\lambda, \phi \rangle_1.$$

Moreover,

$$\begin{aligned} &\left(\chi_{1\lambda} \overline{\varphi_1}' - \chi'_{1\lambda} \overline{\varphi_1} \right) \Big|_{-1}^{h_1} + \frac{1}{\theta} \left(\chi_{2\lambda} \overline{\varphi_2}' - \chi'_{2\lambda} \overline{\varphi_2} \right) \Big|_{h_1}^{h_2} \\ &+ \frac{1}{\theta_\gamma} \left(\chi_{3\lambda} \overline{\varphi_3}' - \chi'_{3\lambda} \overline{\varphi_3} \right) \Big|_{h_2}^{h_3} + \frac{1}{\theta_\gamma \xi} \left(\chi_{4\lambda} \overline{\varphi_4}' - \chi'_{4\lambda} \overline{\varphi_4} \right) \Big|_{h_3}^1 = \end{aligned}$$

$$\begin{aligned}
& \chi_{1\lambda}(h_1, \lambda) \overline{\varphi_1}'(h_1, \lambda) - \chi_{1\lambda}'(h_1, \lambda) \overline{\varphi_1}(h_1, \lambda) - \chi_{1\lambda}(-1, \lambda) \overline{\varphi_1}'(-1, \lambda) + \\
& \chi_{1\lambda}'(-1, \lambda) \overline{\varphi_1}(-1, \lambda) + \frac{1}{\theta} \left(\chi_{2\lambda}(h_2, \lambda) \overline{\varphi_2}'(h_2, \lambda) - \chi_{2\lambda}'(h_2, \lambda) \overline{\varphi_2}(h_2, \lambda) \right. \\
& \left. - \chi_{2\lambda}(h_1, \lambda) \overline{\varphi_2}'(h_1, \lambda) + \chi_{2\lambda}'(h_1, \lambda) \overline{\varphi_2}(h_1, \lambda) \right) + \frac{1}{\theta\gamma} \left(\chi_{3\lambda}(h_3, \lambda) \overline{\varphi_3}'(h_3, \lambda) \right. \\
& \left. - \chi_{3\lambda}'(h_3, \lambda) \overline{\varphi_3}(h_3, \lambda) - \chi_{3\lambda}(h_2, \lambda) \overline{\varphi_3}'(h_2, \lambda) + \chi_{3\lambda}'(h_2, \lambda) \overline{\varphi_3}(h_2, \lambda) \right) + \\
& \frac{1}{\theta\gamma\xi} \left(\chi_{4\lambda}(1, \lambda) \overline{\varphi_4}'(1, \lambda) - \chi_{4\lambda}'(1, \lambda) \overline{\varphi_4}(1, \lambda) - \chi_{4\lambda}(h_3, \lambda) \overline{\varphi_4}'(h_3, \lambda) \right. \\
& \left. + \chi_{4\lambda}'(h_3, \lambda) \overline{\varphi_4}(h_3, \lambda) \right) \\
& = \alpha_1 \chi_{1\lambda}(-1, \lambda) + \alpha_2 \chi_{1\lambda}'(-1, \lambda) + \chi_{1\lambda}(h_1, \lambda) \overline{\varphi_1}'(h_1, \lambda) - \chi_{1\lambda}'(h_1, \lambda) \overline{\varphi_1}(h_1, \lambda) \\
& + \frac{1}{\theta} \left(\chi_{2\lambda}(h_2, \lambda) \overline{\varphi_2}'(h_2, \lambda) - \chi_{2\lambda}'(h_2, \lambda) \overline{\varphi_2}(h_2, \lambda) - \chi_{2\lambda}(h_1, \lambda) \overline{\varphi_2}'(h_1, \lambda) + \right. \\
& \left. \chi_{2\lambda}'(h_1, \lambda) \overline{\varphi_2}(h_1, \lambda) \right) + \frac{1}{\theta\gamma} \left(\chi_{3\lambda}(h_3, \lambda) \overline{\varphi_3}'(h_3, \lambda) - \chi_{3\lambda}'(h_3, \lambda) \overline{\varphi_3}(h_3, \lambda) \right) \\
& - \frac{1}{\theta\gamma} \left(\chi_{3\lambda}(h_2, \lambda) \overline{\varphi_3}'(h_2, \lambda) - \chi_{3\lambda}'(h_2, \lambda) \overline{\varphi_3}(h_2, \lambda) \right) + \\
& \frac{1}{\theta\gamma\xi} \left(\beta_2' \overline{\varphi_4}'(1, \lambda) - \beta_1' \overline{\varphi_4}(1, \lambda) \right) - \frac{1}{\theta\gamma\xi} \left(\chi_{4\lambda}(h_3, \lambda) \overline{\varphi_4}'(h_3, \lambda) - \chi_{4\lambda}'(h_3, \lambda) \overline{\varphi_4}(h_3, \lambda) \right) \\
& = \alpha_1 \chi_{1\lambda}(-1, \lambda) + \alpha_2 \chi_{1\lambda}'(-1, \lambda) + \frac{1}{\theta\gamma\xi} \left(\beta_2' \overline{\varphi_4}'(1, \lambda) - \beta_1' \overline{\varphi_4}(1, \lambda) \right).
\end{aligned}$$

Note that

$$w'(\lambda) = \alpha_2 \chi_{1\lambda}'(-1, \lambda) + \alpha_1 \chi_{1\lambda}(-1, \lambda)$$

Therefore, (9) becomes

$$w'(\lambda) = \langle \chi, \phi \rangle_1 + 2it \langle \chi_\lambda, \phi \rangle_1 - \frac{1}{\theta\gamma\xi} \left(\beta_2' \overline{\varphi_4}'(1, \lambda) - \beta_1' \overline{\varphi_4}(1, \lambda) \right). \quad (10)$$

Now we consider the simplicity of the eigenvalues of (1)–(9). Let μ be arbitrary zero of $w(\lambda)$. By Corollary 1.1, μ is real. Since

$$w(\mu) = \begin{vmatrix} \varphi_1(x, \mu) & \chi_1(x, \mu) \\ \varphi_1'(x, \mu) & \chi_1'(x, \mu) \end{vmatrix} = 0,$$

we have $\varphi_1(x, \mu) = c_1 \chi_1(x, \mu)$ ($c_1 \neq 0$), $\varphi_2(x, \mu) = c_2 \chi_2(x, \mu)$ ($c_2 \neq 0$), $\varphi_3(x, \mu) = c_3 \chi_3(x, \mu)$ ($c_3 \neq 0$) and $\varphi_4(x, \mu) = c_4 \chi_4(x, \mu)$ ($c_4 \neq 0$) where $c_1, c_2, c_3, c_4 \in \mathbb{C}$.

From

$$\begin{aligned}
\varphi_2(h_1, \mu) &= c_1 \left(\alpha_3 \chi_1(h_1, \mu) + \beta_3 \chi_1'(h_1, \mu) \right) = c_1 \chi_2(h_1, \mu), \\
\varphi_2'(h_1, \mu) &= c_1 \left(\alpha_4 \chi_1(h_1, \mu) + \beta_4 \chi_1'(h_1, \mu) \right) = c_1 \chi_2'(h_1, \mu), \\
\varphi_3(h_2, \mu) &= c_2 \left(\alpha_5 \chi_2(h_2, \mu) + \beta_5 \chi_2'(h_2, \mu) \right) = c_2 \chi_3(h_2, \mu), \\
\varphi_3'(h_2, \mu) &= c_2 \left(\alpha_6 \chi_2(h_2, \mu) + \beta_6 \chi_2'(h_2, \mu) \right) = c_2 \chi_3'(h_2, \mu), \\
\varphi_4(h_3, \mu) &= c_3 \left(\alpha_7 \chi_3(h_3, \mu) + \beta_7 \chi_3'(h_3, \mu) \right) = c_3 \chi_4(h_3, \mu), \\
\varphi_4'(h_3, \mu) &= c_3 \left(\alpha_8 \chi_3(h_3, \mu) + \beta_8 \chi_3'(h_3, \mu) \right) = c_3 \chi_4'(h_3, \mu),
\end{aligned}$$

we get $c_1 = c_2 = c_3 = c_4 \neq 0$. Thus, simple calculations using (10) and the initial values of χ_4 at $x = 1$ give

$$\begin{aligned}
w'(\mu) &= \overline{c_1} \left(\frac{1}{p_1^2} \int_{-1}^{h_1} |\chi_1(x, \mu)|^2 dx + \frac{1}{p_2^2 \theta} \int_{h_1}^{h_2} |\chi_2(x, \mu)|^2 dx + \frac{1}{p_3^2 \theta \gamma} \int_{h_2}^{h_3} |\chi_3(x, \mu)|^2 dx \right. \\
&\quad \left. + \frac{1}{p_4^2 \theta \gamma \xi} \int_{h_3}^1 |\chi_4(x, \mu)|^2 dx + \frac{\rho}{\theta \gamma \xi} \right)
\end{aligned}$$

Note that $\rho > 0$, $\theta > 0$, $\gamma > 0$, $\xi > 0$ and $c_1 \neq 0$, so $w'(\mu) \neq 0$. Hence, the analytic multiplicity of μ is one. By Lemma 2.1, the proof is completed. ■

Theorem 2.2 *All eigenvalues of (1) – (9) are geometrically simple.*

Proof. If f and g are two eigenfunctions for an eigenvalue λ_* of (1) – (9), then (2) implies that $f(-1) = Kg(-1)$ and $f'(-1) = Kg'(-1)$ for some constant $K \in \mathbb{R}$. By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (4) – (9), we have that $f = Kg$ on $[-1, 1]$. Thus the geometric multiplicity of λ_* is one. ■

4 Completeness of eigenfunctions

Theorem 3.1 *The operator A has only point spectrum, i.e., $\sigma(A) = \sigma_\rho(A)$.*

Proof. It suffices to prove that if η is not an eigenvalue of A , then $\eta \in \rho(A)$. Since A is self-adjoint, we only consider a real η . We investigate the equation $(A - \eta)Y = F \in H$, where $F = (f, h)$.

Let us consider the initial-value problem

$$\left\{ \begin{array}{l} \tau y - \eta y = f, \quad x \in I, \\ \alpha_1 y(-1) + \alpha_2 y'(-1) = 0, \\ y(h_1 + 0) = \alpha_3 y(h_1 - 0) + \beta_3 y'(h_1 - 0), \\ y'(h_1 + 0) = \alpha_4 y(h_1 - 0) + \beta_4 y'(h_1 - 0), \\ y(h_2 + 0) = \alpha_5 y(h_2 - 0) + \beta_5 y'(h_2 - 0), \\ y'(h_2 + 0) = \alpha_6 y(h_2 - 0) + \beta_6 y'(h_2 - 0), \\ y(h_3 + 0) = \alpha_7 y(h_3 - 0) + \beta_7 y'(h_3 - 0), \\ y'(h_3 + 0) = \alpha_8 y(h_3 - 0) + \beta_8 y'(h_3 - 0), \end{array} \right. \quad (11)$$

Let $u(x)$ be the solution of the equation $\tau u - \eta u = 0$ satisfying

$$\begin{aligned} u(-1) &= \alpha_2, \quad u'(-1) = -\alpha_1, \\ u(h_1 + 0) &= \alpha_3 u(h_1 - 0) + \beta_3 u'(h_1 - 0), \\ u'(h_1 + 0) &= \alpha_4 u(h_1 - 0) + \beta_4 u'(h_1 - 0), \\ u(h_2 + 0) &= \alpha_5 u(h_2 - 0) + \beta_5 u'(h_2 - 0), \\ u'(h_2 + 0) &= \alpha_6 u(h_2 - 0) + \beta_6 u'(h_2 - 0), \\ u(h_3 + 0) &= \alpha_7 u(h_3 - 0) + \beta_7 u'(h_3 - 0), \\ u'(h_3 + 0) &= \alpha_8 u(h_3 - 0) + \beta_8 u'(h_3 - 0). \end{aligned}$$

In fact,

$$u(x) = \left\{ \begin{array}{ll} u_1(x), & x \in [-1, h_1), \\ u_2(x), & x \in (h_1, h_2), \\ u_3(x), & x \in (h_2, h_3), \\ u_4(x), & x \in (h_3, 1], \end{array} \right.$$

where $u_1(x)$ is the unique solution of the initial-value problem

$$\left\{ \begin{array}{l} -p_1^2 u'' + q(x)u = \eta u, \quad x \in [-1, h_1), \\ u(-1) = \alpha_2, \quad u'(-1) = -\alpha_1; \end{array} \right.$$

$u_2(x)$ is the unique solution of the problem

$$\left\{ \begin{array}{l} -p_2^2 u'' + q(x)u = \eta u, \quad x \in (h_1, h_2), \\ u_2(h_1) = \alpha_3 u_1(h_1) + \beta_3 u_1'(h_1), \\ u_2'(h_1) = \alpha_4 u_1(h_1) + \beta_4 u_1'(h_1); \end{array} \right.$$

$u_3(x)$ is the unique solution of the problem

$$\left\{ \begin{array}{l} -p_3^2 u'' + q(x)u = \eta u, \quad x \in (h_2, h_3), \\ u_3(h_2) = \alpha_5 u_2(h_2) + \beta_5 u_2'(h_2), \\ u_3'(h_2) = \alpha_6 u_2(h_2) + \beta_6 u_2'(h_2). \end{array} \right.$$

and $u_4(x)$ is the unique solution of the problem

$$\begin{cases} -p_4^2 u'' + q(x)u = \eta u, & x \in (h_3, 1], \\ u_3(h_3) = \alpha_7 u_3(h_3) + \beta_7 u_3'(h_3), \\ u_3'(h_3) = \alpha_8 u_3(h_3) + \beta_8 u_3'(h_3). \end{cases}$$

Let

$$w(x) = \begin{cases} w_1(x), & x \in [-1, h_1), \\ w_2(x), & x \in (h_1, h_2), \\ w_3(x), & x \in (h_2, h_3), \\ w_4(x), & x \in (h_3, 1], \end{cases}$$

be a solution of $\tau w - \eta w = f$ satisfying

$$\begin{aligned} \alpha_1 w(-1) + \alpha_2 w'(-1) &= 0, \\ w(h_1+0) &= \alpha_3 w(h_1-0) + \beta_3 w'(h_1-0), \\ w'(h_1+0) &= \alpha_4 w(h_1-0) + \beta_4 w'(h_1-0), \\ w(h_2+0) &= \alpha_5 w(h_2-0) + \beta_5 w'(h_2-0), \\ w'(h_2+0) &= \alpha_6 w(h_2-0) + \beta_6 w'(h_2-0), \\ w(h_3+0) &= \alpha_7 w(h_3-0) + \beta_7 w'(h_3-0), \\ w'(h_3+0) &= \alpha_8 w(h_3-0) + \beta_8 w'(h_3-0). \end{aligned}$$

Then, (11) has the general solution

$$y(x) = \begin{cases} du_1 + w_1, & x \in [-1, h_1), \\ du_2 + w_2, & x \in (h_1, h_2), \\ du_3 + w_3, & x \in (h_2, h_3), \\ du_4 + w_4, & x \in (h_3, 1], \end{cases} \quad (12)$$

where $d \in \mathbb{C}$.

Since η is not an eigenvalue of the problem (1) – (7), we have

$$\eta [\beta_1' u_2(1) - \beta_2' u_2'(1)] + [\beta_1 u_2(1) - \beta_2 u_2'(1)] \neq 0. \quad (13)$$

The second component of $(A - \eta)Y = F$ involves the equation

$$-N(y) - \eta N'(y) = h,$$

namely,

$$[-\beta_1 y(1) + \beta_2 y'(1)] - \eta [\beta_1' y(1) - \beta_2' y'(1)] = h. \quad (14)$$

Substituting (12) into (14), we get

$$\begin{aligned} & (\beta_2 u_2'(1) - \beta_1 u_2(1) + \eta \beta_2' u_2'(1) - \eta \beta_1' u_2(1)) d \\ & = h + \beta_1 w_2(1) - \beta_2 w_2'(1) + \eta \beta_1' w_2(1) - \eta \beta_2' w_2'(1) \end{aligned}$$

In view of (13), we know that d is uniquely solvable. Therefore, y is uniquely determined.

The above arguments show that $(A - \eta I)^{-1}$ is defined on all of H , where I is identity matrix. We obtain that $(A - \eta I)^{-1}$ is bounded by Theorem 1.1 and the Closed Graph Theorem. Thus, $\eta \in \rho(A)$. Therefore, $\sigma(A) = \sigma_\rho(A)$. ■

Lemma 3.1 [10] *The eigenvalues of the boundary value problem (1) – (9) are bounded below, and they are countably infinite and can cluster only at ∞ .*

For every $\delta \in \mathbb{R} \setminus \sigma_\rho(A)$, we have the following immediate conclusion.

Lemma 3.2 *Let λ be an eigenvalue of $A - \delta I$, and V a corresponding eigenfunction. Then, λ^{-1} is an eigenvalue of $(A - \delta I)^{-1}$, and V is a corresponding eigenfunction. The converse is also true.*

On the other hand, if μ is an eigenvalue of A and U is a corresponding eigenfunction, then $\mu - \delta$ is an eigenvalue of $A - \delta I$, and U is a corresponding eigenfunction. The converse is also true. Accordingly, the discussion about the completeness of the eigenfunctions of A is equivalent to considering the corresponding property of $(A - \delta I)^{-1}$.

By Lemma 1.1, Lemma 3.1 and Corollary 1.1, we suppose that $\{\lambda_n; n \in \mathbb{N}\}$ is the real sequence of eigenvalues of A , then $\{\lambda_n - \delta; n \in \mathbb{N}\}$ is the sequence of eigenvalues of $A - \delta I$. We may assume that

$$|\lambda_1 - \delta| \leq |\lambda_2 - \delta| \leq \dots \leq |\lambda_n - \delta| \leq \dots \rightarrow \infty.$$

Let $\{\mu_n; n \in \mathbb{N}\}$ be the sequence of eigenvalues of $(A - \delta I)^{-1}$. Then $\mu_n = (\lambda_n - \delta)^{-1}$ and

$$|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n| \geq \dots \rightarrow 0.$$

Note that 0 is not an eigenvalue of $(A - \delta I)^{-1}$.

Theorem 3.2 *The operator A has compact resolvents, i.e., for each $\delta \in \mathbb{R} \setminus \sigma_\rho(A)$, $(A - \delta I)^{-1}$ is compact on H .*

Proof. Let $\{\mu_1, \mu_2, \dots\}$ be the eigenvalues of $(A - \delta I)^{-1}$, and $\{P_1, P_2, \dots\}$ the orthogonal projections of finite rank onto the corresponding eigenspaces. Since $\{\mu_1, \mu_2, \dots\}$ is a bounded sequence and all P_n 's are mutually orthogonal, we have $\sum_{n=1}^{\infty} \mu_n P_n$ is strongly convergent to the bounded operator $(A - \delta I)^{-1}$, i.e., $(A - \delta I)^{-1} = \sum_{n=1}^{\infty} \mu_n P_n$. Because for every $\alpha > 0$, the number of μ_n 's satisfying $|\mu_n| > \alpha$ is finite, and all P_n 's are of finite rank, we obtain that $(A - \delta I)^{-1}$ is compact. ■

In terms of the above statements and the spectral theorem for compact operators, we obtain the following theorem.

Theorem 3.3 *The eigenfunctions of the problem (1) – (9), augmented to become eigenfunctions of A , are complete in H , i.e., if we let*

$$\left\{ \Phi_n = \left(\phi_n(x), N'(\phi_n) \right); n \in \mathbb{N} \right\}$$

be a maximum set of orthonormal eigenfunctions of A , where $\{\phi_n(x); n \in \mathbb{N}\}$ are eigenfunctions of (1) – (9), then for all $F \in H$, $F = \sum_{n=1}^{\infty} \langle F, \Phi_n \rangle \Phi_n$.

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